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Similarity of the two-dimensional and axisymmetric boundary-layer flows for purely viscous non-Newtonian fluids

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Conditions for the existence of similar solutions for the two-dimensional and axisymmetric boundary layers are obtained for the steady or unsteady flow for purely viscous non-Newtonian fluids, where the shear stress is proportional to the nth power of the velocity gradient and $n > 0$. These conditions are shown to be a generalization of the similarity conditions for Newtonian fluids. In particular it is found that the velocity at the outer edge of the boundary layer must be proportional to $\{R(x)+A\}^m$ or $\exp[AR(x)]$ for steady flows and to $(t+A)^m$, $(dx+A)/(t+B)$ or $\exp[At]$ for unsteady flows. Here x is the distance along the wall from the forward stagnation point, t is the time, $R(x) = x$ for two-dimensional flows and $R(x) = \int r^{1+n}dx$ for axisymmetric flows, r is the distance from the axis to the wall, and A, B, d, m are constants.

Several examples of the similar solutions are calculated analytically for both steady and unsteady pseudoplastic flows ($n < 1$). In these solutions the velocity in the boundary layer tends to the outside velocity in such a manner that the difference tends to zero as an inverse power of the distance from the wall, whereas such a difference tends to zero exponentially for the corresponding flow in Newtonian fluids.

1. Introduction

Many exact solutions for the boundary-layer equations have been obtained for the flow for Newtonian fluids.* Some of them are similar or affine solutions, for which a component of the velocity has such a property that two velocity profiles at different co-ordinates differ only by a scale factor. In general, it is assumed that the radius of curvature of the wall is far greater than the thickness of the boundary layer. Similar solutions for two-dimensional and axisymmetric flows have been investigated thoroughly for steady flows (Hayasi 1960) and unsteady flows (Hayasi 1961).

Recently some examples of the two-dimensional similar boundary layer have been investigated for non-Newtonian fluids, where the shear stress is proportional to the nth power of the velocity gradient and $n > 0$. Schowalter (1960) considered conditions for the existence of similar solutions for steady flows. His treatment, however, does not cover all cases (see § 6). Acrivos *et al.* (1960)

* For these solutions the reader is referred to literatures by Schlichting (1960), Hayasi (1960, 1961, 1962), and Curle (1962).

calculated numerically the velocity distributions in the boundary layer on a semi-infinite flat plate. Bird (1959) and Wells (1964*a*) considered the unsteady boundary layer for the so-called Rayleigh problem. Wells (1964*b*) investigated the conditions for the existence of similar solutions for both steady and unsteady flows. Unfortunately his treatment is neither self-evident nor accurate (see § 6). Recently it was found that there is a correlation of steady two-dimensional and axisymmetric boundary-layer flows, and that this suggests similar boundary layers exist also for steady axisymmetric flows (Hayasi 1965). Therefore, we wish to extend our previous analyses to non-Newtonian fluids, considering both two-dimensional and axisymmetric flows.

2. Fundamental equations

Using the co-ordinates (x, y), in which x is the distance along the wall from the forward stagnation point and y is the distance from the wall, the equations for the two-dimensional and axisymmetric boundary layers in purely viscous incompressible non-Newtonian fluids can be written as

$$\partial(ru)/\partial x + \partial(rv)/\partial y = 0, \tag{2.1}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}, \tag{2.2}$$

$$\partial p/\partial y = 0, \tag{2.3}$$

where $r = 1$ for two-dimensional flow and r is the distance from the axis to the wall for axisymmetric flow, u and v are components of the velocity in the directions of x and y, respectively, t the time, ρ the density, p the pressure, and τ_{yx} is the shear stress in the x-direction due to the velocity gradient in the y-direction. The boundary conditions for u and v are

$$\left.\begin{aligned} u = v = 0 \quad &\text{at} \quad y = 0, \\ u = U(x,t) \quad &\text{for} \quad y \to \infty. \end{aligned}\right\} \tag{2.4}$$

From Bernoulli's theorem, we have

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x}. \tag{2.5}$$

It has been shown by Schowalter (1960) that for two-dimensional flow the shear stress can be expressed as

$$\tau_{yx} = a(\partial u/\partial y)^n, \quad (n > 0), \tag{2.6}$$

where a and n are constants, provided that the velocity gradient is always non-negative, and that the Ostwald–de Waele (power-law) model is adopted. It can be easily shown that this equation is also valid for axisymmetric flow. Substituting (2.5) and (2.6) into (2.2), we obtain

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = \frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+\nu n\left(\frac{\partial u}{\partial y}\right)^{n-1}\frac{\partial^2 u}{\partial y^2}, \tag{2.7}$$

where $\nu = a/\rho$. Equations (2.1), (2.3), and (2.7) are fundamental equations for the present problem.

It has been shown by Acrivos *et al.* (1960) that when $n > 2$ the boundary-layer flows are not of much practical interest, since Reynolds number for a power law non-Newtonian fluid is given by

$$\mathrm{Re} = U_1^{2-n}L^n/\nu,$$

where U_1 and L are characteristic velocity and length, respectively, so that range of validity of the boundary-layer assumption appears to be rather limited.

3. Conditions for the existence of similar solutions

In this section we consider the conditions under which the boundary-layer flow is similar. For this purpose we assume

$$u = U(x,t)f'(\eta), \tag{3.1}$$

where

$$\eta = y/\nu^{1/(1+n)}g(x,t) \tag{3.2}$$

is a non-dimensional variable, and primes denote differentiation. Naturally g is considered to be proportional to the thickness of the boundary layer. The boundary conditions for f are

$$f(0) = f'(0) = 0; \quad f'(\infty) = 1. \tag{3.3}$$

Integrating (2.1) with respect to y and using (3.1) and (3.3), we obtain

$$\nu^{-1/(1+n)}\frac{v}{U} = -f\frac{g}{rU}\frac{\partial(rU)}{\partial x} + (\eta f' - f)\frac{\partial g}{\partial x}. \tag{3.4}$$

Substituting (3.4) into (2.7) gives

$$n\frac{U^{n-1}}{g^{1+n}}(f'')^{n-1}f''' + \left(\frac{U}{r}\frac{dr}{dx} + \frac{\partial U}{\partial x} + \frac{U}{g}\frac{\partial g}{\partial x}\right)ff'' + \frac{\partial U}{\partial x}(1-f'^2) + \frac{1}{g}\frac{\partial g}{\partial t}\eta f'' + \frac{1}{U}\frac{\partial U}{\partial t}(1-f') = 0. \tag{3.5}$$

Hence we must have

$$\frac{1}{U^n}\frac{\partial U}{\partial t} = \frac{a}{g^{1+n}}, \tag{3.6a}$$

$$g^n\frac{\partial g}{\partial t} = bU^{n-1}, \tag{3.6b}$$

$$U^{1-n}\frac{\partial U}{\partial x} = \frac{d}{g^{1+n}}, \tag{3.6c}$$

$$\frac{1}{g}\frac{\partial g}{\partial x} + \frac{1}{r}\frac{dr}{dx} = e\frac{U^{n-2}}{g^{1+n}}, \tag{3.6d}$$

a, b, d, and e being certain non-dimensional constants. Then equation (3.5) is reduced to

$$n(f'')^{n-1}f''' + (d+e)ff'' + d(1-f'^2) + b\eta f'' + a(1-f') = 0. \tag{3.7}$$

Differentiating (3.6a) with respect to x and use of (3.6b) and (3.6c) give, after some transformations,

$$\{a + (1+n)b\}d = (1+n)aU^{2-n}g^n\partial g/\partial x. \tag{3.8}$$

Similarly we have from (3.6b)

$$\{(2-n)a+(1+n)b\}e+(n-1)bd = (1+n)bU^{2-n}g^n\,\partial g/\partial x. \tag{3.9}$$

Differentiating (3.8) with respect to t and employing (3.6a), (3.6b) and (3.6c), we have

$$a\{(2-n)aU^{2-n}g^n\,\partial g/\partial x+(n-1)bd\} = 0.$$

If $a = 0$, however, from (3.8) it is clear that $bd = 0$. Thus we obtain

$$(2-n)aU^{2-n}g^n\,\partial g/\partial x+(n-1)bd = 0, \tag{3.10}$$

regardless of the value of a. Use of (3.8) yields

$$\left(\frac{2-n}{1+n}a+b\right)d = 0. \tag{3.11}$$

Eliminating $\partial g/\partial x$ from (3.8) and (3.9), and using (3.11), we have

$$\left(\frac{2-n}{1+n}a+b\right)ae = 0. \tag{3.12}$$

From these relations we can conclude, after some simple considerations, that there are only the following possibilities.

(1) *The case* $a = 0$

As mentioned above, we have $bd = 0$.

(1.1) *The case* $b = 0$. This case corresponds to the steady flow.

(1.1.1) *The case* $d = 0$

$$\left.\begin{aligned}&U = \text{const.},\\ &g = r^{-1}\{(1+n)eU^{n-2}R(x)+A\}^{1/(1+n)},\end{aligned}\right\} \tag{3.13a}$$

$$nf'''+ef(f'')^{2-n} = 0, \tag{3.13b}$$

where $R(x) = \int r^{1+n}dx$ and A is an integration constant.

(1.1.2) *The case* $d \neq 0$

Eliminating g from (3.6c) and (3.6d) gives

$$U^{(1/m)-1}dU/dx = A\,dr^{1+n},$$

where $A \neq 0$ and $1/m = (1+n)(e/d)+2-n$;

$$n(f'')^{n-1}f'''+(d+e)ff''+d(1-f'^2) = 0. \tag{3.14}$$

(1.1.2.1) *The case* $1/m = 0$

$$\left.\begin{aligned}&e = -(2-n)d/(1+n), \quad U = B\exp[AdR(x)],\\ &g = \frac{1}{r}(AB^{2-n})^{-1/(1+n)}\exp\left\{\frac{n-2}{1+n}A\,dR(x)\right\} = \frac{1}{r}(AU^{2-n})^{-1/(1+n)},\end{aligned}\right\} \tag{3.15}$$

where B is another integration constant.

(1.1.2.2) *The case* $1/m \neq 0$

$$\left.\begin{aligned}&U = m^{-m}\{A\,dR(x)+B\}^m,\\ &g = \frac{1}{r}A^{-1/(1+n)}[\{A\,dR(x)+B\}/m]^{me/d} = \frac{1}{r}A^{-1/(1+n)}U^{e/d}.\end{aligned}\right\} \tag{3.16}$$

(1.2) *The case* $b \neq 0$

$$\left.\begin{aligned} d &= 0, \\ U &= \text{const.}, \quad r = \text{const.}, \\ g &= \{(1+n)\,eU^{n-2}x+(1+n)\,bU^{n-1}t+A\}^{1/(1+n)}, \end{aligned}\right\} \tag{3.17a}$$

$$nf''''+(ef+b\eta)(f'')^{2-n} = 0. \tag{3.17b}$$

The form of the wall allowing such similar solutions must be a cylinder.

(2) *The case* $a \neq 0$

$$\left(\frac{2-n}{1+n}a+b\right)d = \left(\frac{2-n}{1+n}a+b\right)e = 0.$$

(2.1) *The case* $(2-n)\,a+(1+n)\,b = 0$,

$$\left.\begin{aligned} b = (n-2)a/(1+n), \quad U &= -(dx+B)/(at+A), \\ g &= (dx+B)^{(n-1)/(1+n)}(-at-A)^{(2-n)/(1+n)}, \end{aligned}\right\} \tag{3.18a}$$

$$n(f'')^{n-1}f'''+(d+e)ff''+d(1-f'^2)-\frac{2-n}{1+n}a\eta f''+a(1-f') = 0. \tag{3.18b}$$

(2.1.1) *The case* $d = 0$

$$r = C\exp\,[ex/B], \tag{3.18c}$$

where C is another integration constant.

(2.1.2) *The case* $d \neq 0$

$$r = C(dx+B)^{\frac{1-n}{1+n}+\frac{e}{d}}. \tag{3.18d}$$

(2.2) *The case* $(2-n)\,a+(1+n)\,b \neq 0$,

$$d = e = 0, \quad r = \text{const.}, \quad v = 0, \tag{3.19a}$$

$$n(f'')^{n-1}f'''+b\eta f''+a(1-f') = 0. \tag{3.19b}$$

(2.2.1) *The case* $q = (1+n)\,(b/a)+1-n = 0$

$$U = B\exp\,[at/A], \quad g = A^{1/(1+n)}B^{b/a}\exp\,[bt/A]. \tag{3.20}$$

(2.2.2) *The case* $q \neq 0$

$$U = A\{q(at+B)\}^{1/q}, \quad g = A^{(n-1)/(1+n)}\{q(at+B)\}^{b/aq}. \tag{3.21}$$

In these equations constants a, b, d, and e might not always be able to take arbitrary values for a solution or solutions satisfying the boundary conditions to exist. For example, if we assume $e = 0$ in (3.13b), it is clear that the required solution does not exist.

It should be noted for two-dimensional flows that, when $g = g(t)$, we have

$$\partial^2u/\partial x^2 = f'\,\partial^2U/\partial x^2 = 0$$

except for the case (1.1.2.1), so that equation (3.7) is the same as what should be reduced from the Navier–Stokes equation.

4. An example of the steady similar boundary layer

In this section we show an example of the steady similar boundary layer. For this purpose we choose the case (1.1.2.1) of §3, where $a = b = 0$, and assume that n is equal to $\frac{1}{2}$ and $d > 0$. Then we have $d+e = 0$, so that equation (3.14) is reduced to

$$(\phi')^{\frac{1}{2}}\phi'' - (\phi^2 - 1)\,\phi' = 0, \tag{4.1}$$

where $\phi = df/d\eta = u/U$ and primes denote differentiation with respect to

$$\xi = (2d)^{\frac{2}{3}}\eta.$$

Boundary conditions are given by

$$\phi(0) = 0, \quad \phi(\infty) = 1. \tag{4.2}$$

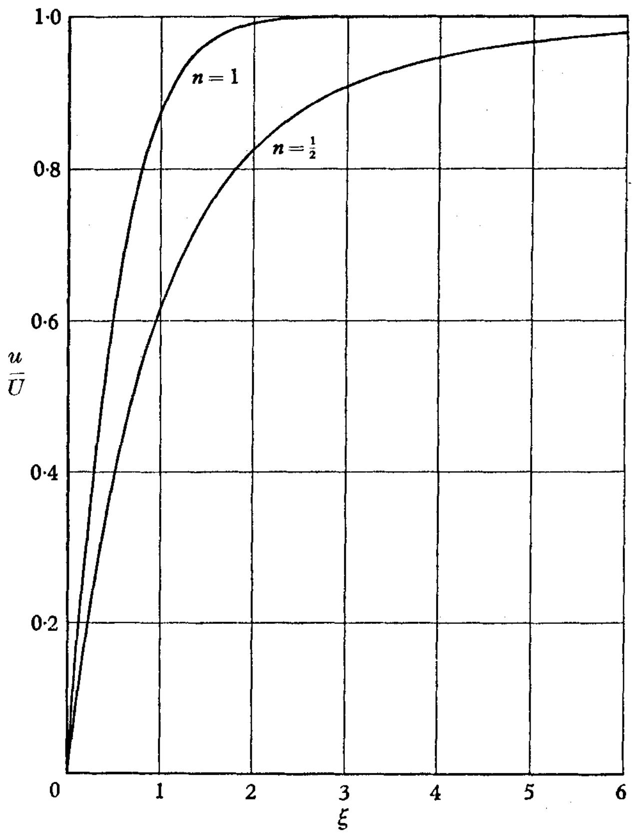

FIGURE 1. Steady similar velocity profile when $a = b = d+e = 0$, $d > 0$. (The profile for $n = 1$ is given by Hartree 1937.)

Equation (4.1) can be integrated as

$$\phi' = 2^{-\frac{2}{3}}(\phi^3 - 3\phi + A)^{\frac{2}{3}},$$

where A is an integration constant and $A = 2$, since $\phi \to 1$ and $\phi' \to 0$ when $\xi \to \infty$. We can integrate this equation again, getting

$$\left(\frac{2+\phi}{1-\phi}\right)^{\frac{1}{3}} = 2^{-\frac{2}{3}}\xi + B,$$

where B is another integration constant. Since $\phi(0) = 0$, we have $B = 2^{\frac{1}{3}}$. Thus we obtain

$$\phi = \frac{u}{U} = \frac{(\xi+2)^3-8}{(\xi+2)^3+4}. \tag{4.3}$$

Therefore $U-u$ tends to zero as η^{-3} when $\eta \to \infty$. Such a solution for which $U-u$ tends to zero in the order of negative powers of η as $\eta \to \infty$ has been called a 'weak solution' (Hayasi 1961).

From (3.4) and (3.15), we have

$$v = -\xi\phi\left(\frac{\nu}{2Ad}\right)^{\frac{2}{3}}\left(A\,dr^{\frac{1}{3}}+\frac{1}{r^2}\frac{dr}{dx}\right). \tag{4.4}$$

The velocity distribution u/U is presented in figure 1. This problem for $n = 1$ was treated by Goldstein (1939), and Hartree (1937)* had given the 'medium solution' for which $U-u$ tends to zero as $\exp[-\eta]$. This solution is also included in the figure. It is noted that $\xi = (d/2)^{\frac{1}{2}}\eta$ for $n = 1$.

5. Examples of the unsteady similar boundary layer

In this section we show examples of the unsteady similar boundary layer. First we consider the case (1.2) of §3, where $a = d = 0$, and assume $e = 0$ and $b > 0$. Then equation (3.17b) reduces to

$$\phi'' + 2\xi(\phi')^{2-n} = 0, \tag{5.1}$$

where $\phi = df/d\eta = u/U$ as before, and primes denote differentiation with respect to $\xi = (b/2n)^{1/(1+n)}\eta$. Boundary conditions are given by (4.2). From (3.4) and (3.17a) it is clear that $v = 0$. Equation (5.1) is integrated once to give

$$\phi' = \{A+(1-n)\,\xi^2\}^{-1/(1-n)} \tag{5.2}$$

for $n \neq 1$. Solutions of this equation have been calculated numerically by Wells (1964a) for $n = \frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{5}{7}$ and $\frac{7}{9}$.

This case corresponds to the Rayleigh problem in which relative motion of a fluid and of an immersed solid body started impulsively from rest is considered. Using a co-ordinate system fixed in the fluid, Bird (1959) presented analytical solutions. Using the co-ordinate system fixed to the wall, his results are re-written as follows:

$$\phi = \frac{u}{U} = \frac{\xi}{(\xi^2+\alpha)^{\frac{1}{2}}}, \qquad \alpha = \tfrac{3}{2}A = (\tfrac{3}{2})^{\frac{3}{2}} = 1{\cdot}8371, \qquad (n = \tfrac{1}{3})$$

$$\phi = \frac{2}{\pi}\left(\frac{\alpha\xi}{\xi^2+\alpha^2}+\tan^{-1}\frac{\xi}{\alpha}\right), \qquad \alpha = (2A)^{\frac{1}{2}} = \pi^{\frac{1}{2}} = 1{\cdot}4646, \qquad (n = \tfrac{1}{2})$$

$$\phi = \frac{2}{\pi}\left\{\frac{\alpha\xi(3\xi^2+5\alpha^2)}{3(\xi^2+\alpha^2)^2}+\tan^{-1}\frac{\xi}{\alpha}\right\}, \qquad \alpha = (3A)^{\frac{1}{2}} = (\tfrac{3}{2})^{\frac{1}{5}}\pi^{\frac{1}{5}} = 1{\cdot}7390, \qquad (n = \tfrac{2}{3})$$

$$\phi = \frac{2}{\pi}\left\{\frac{128\alpha^9\xi}{315(\xi^2+\alpha^2)^5}+\frac{16\alpha^7\xi}{35(\xi^2+\alpha^2)^4}+\frac{8\alpha^5\xi}{15(\xi^2+\alpha^2)^3}+\frac{2\alpha^3\xi}{3(\xi^2+\alpha^2)^2}+\frac{\alpha\xi}{\xi^2+\alpha^2}+\tan^{-1}\frac{\xi}{\alpha}\right\},$$

$$\alpha = (6A)^{\frac{1}{2}} = \left(\frac{45927\pi}{8}\right)^{\frac{1}{11}} = 2{\cdot}4374. \qquad (n = \tfrac{5}{6})$$

* There is a misprint in his table. The numerical value for ϕ (0·8) should be read as 0·7958 instead of 0·7858.

For Newtonian fluids ($n = 1$), Blasius (1908) obtained the solution

$$\phi = \operatorname{erf}\xi = 2\pi^{-\frac{1}{2}}\int_0^{\xi}\exp(-\xi^2)\,d\xi.$$

For $n > 1$, it seems to be impossible to obtain the solution of (5.2) satisfying (4.2). For example, if we put $n = \frac{3}{2}$, the general solution of (5.2) is

$$\phi = \tfrac{1}{20}\xi^5 - \tfrac{1}{3}A\xi^3 + A^2\xi + B,$$

and a boundary condition $\phi(\infty) = 1$ cannot be satisfied.

From the analytical solutions, it can be easily shown that $U-u$ tends to zero as η^{-m} for $n < 1$, where $m = 2, 3, 5$, and 11 for $n = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$, and $\frac{5}{6}$, respectively. Thus the solutions for pseudoplastic flow ($n < 1$) are weak solutions. It is well known that the Blasius solution for Newtonian flow ($n = 1$) is a 'strong solution' which tends to zero as $\exp(-\eta^2)$ when $\eta \to \infty$.

Next we consider the case (2.2.2) of §3, where $d = e = 0$, and assume $b = 0$ and $a > 0$. Then equation (3.19b) reduces to

$$\tfrac{1}{2}(1+n)(\phi')^{n-1}\phi'' - \phi + 1 = 0, \tag{5.3}$$

where primes denote differentiation with respect to

$$\xi = \{(1+n)a/(2n)\}^{1/(1+n)}\eta.$$

Integration of this equation gives

$$(\phi')^{1+n} = \phi^2 - 2\phi + A,$$

where an integration constant A is determined as unity, since $\phi' \to 0$ when $\phi \to 1$. Integrating once more gives

$$\{(1+n)/(1-n)\}(1-\phi)^{-(1-n)/(1+n)} = \xi + B$$

for $n \neq 1$. Since $\phi(0) = 0$, we obtain $B = (1+n)/(1-n)$. Thus we get

$$\phi = \frac{u}{U} = 1 - \left(\frac{1-n}{1+n}\xi + 1\right)^{-(1+n)/(1-n)}. \tag{5.4}$$

Clearly we must have $n < 1$ for this solution to tend to unity when $\xi \to \infty$. For a Newtonian fluid ($n = 1$), the 'medium' solution of equation (5.3) has been given by Schuh (1955) as

$$\phi = 1 - \exp[-\xi]. \tag{5.5}$$

It is easy to show that equation (5.5) is the limiting form of equation (5.4) when $n \to 1$. Velocity distributions $\phi = u/U$ are presented in figure 2 for $n = \frac{1}{9}, \frac{1}{5}, \frac{1}{3}, \frac{1}{2}$ and 1.

It should be noted that for both cases there is a weak solution in which $U-u$ tends to zero from above as $\eta^{-(1+n)/(1-n)}$ for $0 < n < 1$.

It is noted that for the two-dimensional case equations (5.2) and (5.3) are the same as those to which the Navier–Stokes equation should be reduced, since $g = g(t)$ in these cases.

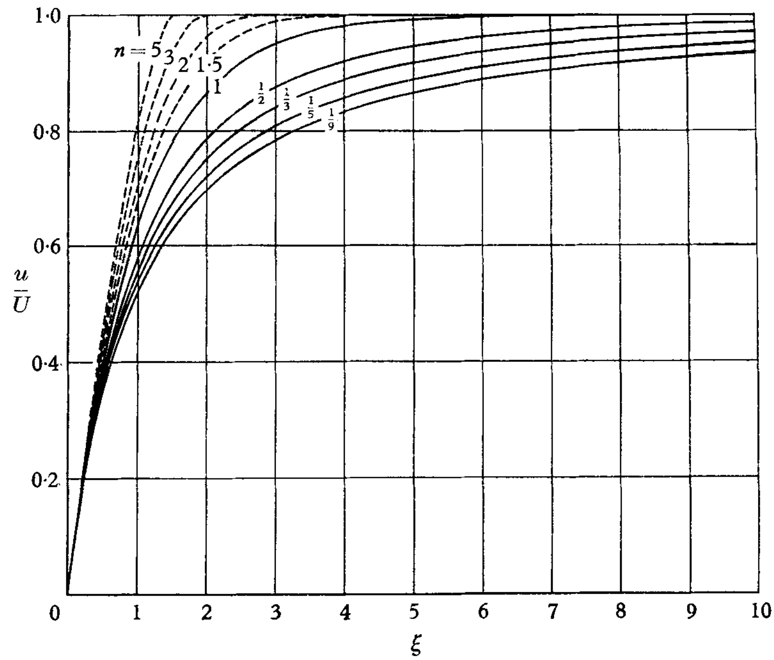

FIGURE 2. Unsteady similar velocity profile when $b = d = e = 0$, $a > 0$. (The profile for $n = 1$ is given by Schuh 1955.)

6. Discussion

For the steady two-dimensional flow, Schowalter (1960) obtained conditions for the existence of similar solutions. He classifies flows into the following three types:

(1) the case $(2-n)d+(1+n)e \neq 0$ and $d+e \neq 0$;

(2) the case $(2-n)d+(1+n)e \neq 0$ and $d+e = 0$;

and (3) the case $(2-n)d+(1+n)e = 0$ and $d+e \neq 0$.

It is evident that there is another case where $(2-n)d+(1+n)e = 0$ and $d+e = 0$. In this case we have $n = \frac{1}{2}$, and this was considered in detail in §4.

For the steady and unsteady two-dimensional flows, Wells (1964*b*) derived conditions for the existence of similar solutions. His treatment is not self-evident; for example, it is not clear whether conditions (3.10), (3.11), and (3.12) are taken into account. Moreover there are nearly one hundred misprints and some of the final results are also erroneous. For example, for the case (1.1.2.1) of §3 of the present paper, using (3.15), equation (3.14) is reduced to

$$(f'')^{n-1}f''' + \frac{d}{n}\left(\frac{2n-1}{1+n}ff'' - f'^2 + 1\right) = 0.$$

This equation corresponds to his equation (31) where he writes $\frac{1}{2}$ in place of $(2n-1)/(1+n)$. Clearly such simplification cannot be justified for non-Newtonian fluid ($n \neq 1$).

It may be interesting to note that the investigation of the asymptotic behaviour of the similar solutions for Newtonian fluids (Hayasi 1961) can be easily extended so as to include the asymptotic behaviour for non-Newtonian fluids. The results are as follows. (1) If $b+d+e = -(a+2d) \neq 0$, or if $b+e \neq 0$ and $a = d = 0$, there may be a 'weak' solution or solutions for $0 < n < 1$ and $(b+d+e)^{1/(1-n)} > 0$, and there is no required solution for $1 < n$ or for $0 < n < 1$ and $(b+d+e)^{1/(1-n)} < 0$; (2) If $b = d+e = 0$ and $a+2d \neq 0$, there may be a 'weak' solution or solutions for $0 < n < 1$ and $a+2d > 0$, and there is no solution for $1 < n$ or for $0 < n < 1$ and $a+2d < 0$; (3) For the weak solution $U-u$ tends to zero from above as $\eta^{-(1+n)/(1-n)}$. Thus for such cases solutions for non-Newtonian fluids exist only for pseudoplastic flows ($n < 1$). These results agree completely with the analytic solutions obtained in §§4 and 5.

For the flow past a semi-infinite flat plate at zero angle of attack or a hollow circular cylinder whose axis is parallel to the stream, we have $U =$ const. so that the thickness of the boundary layer is given by the second equation of (3.13*a*), where $e > 0$. Here we can make $A = 0$ by suitable choice of the coordinates. Equation (3.13*b*) reduces to

$$n(n+1)F''' + F(F'')^{2-n} = 0, \tag{6.1}$$

where $F = \{(1+n)e\}^{1/(1+n)}f$ and primes denote differentiation with respect to $\xi = \{(1+n)e\}^{1/(1+n)}\eta$. Boundary conditions are

$$F(0) = F'(0) = 0; \quad F'(\infty) = 1. \tag{6.2}$$

As is well known Blasius (1908) calculated the solution for a Newtonian fluid ($n = 1$). Acrivos *et al.* (1960) obtained solutions for $n = 0{\cdot}1$, $0{\cdot}2$, $0{\cdot}5$, and $1{\cdot}5$ by numerical calculation, and found that there is no solution for $n \geqslant 2$. From the results in the previous paragraph, however, for $a = b = d = 0$ and $e > 0$ there may be a weak solution or solutions for $0 < n < 1$ but there is no solution for $1 < n$. It may be suspected that their solution for $n = 1{\cdot}5$ does not satisfy a boundary condition $F'(\infty) = 1$.*

So for the power-law assumption (2.6) has been used, although this is valid only if the velocity gradient $\partial u/\partial y$ is relatively large. Now it is clear that $\partial u/\partial y$ is small near the outer edge of the boundary layer, so that the fundamental equation (2.7) is no longer valid in this region. This may be the reason why there is no solution satisfying the boundary condition at $y = \infty$ for dilatant flows ($n > 1$) and there are only weak solutions for pseudoplastic flows ($n < 1$). Acrivos *et al.* (1960) propose to replace the boundary conditions (3.3) by other boundary conditions

$$f(0) = f'(0) = 0, \quad f'(C) = 1, \quad f''(C) = 0, \tag{6.3}$$

where C is a constant to be determined by these equations, provided the former cannot be satisfied. Solutions of (5.4) for $n > 1$ satisfying (6.3) are included in figure 2 as broken lines, although the range of validity of solutions for $n > 2$ is rather limited. From this figure it seems to be plausible to use (6.3) for $n > 1$, though further investigation is needed.

* After the completion of this work the author was informed from Professor Acrivos that this is the case.

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